Algebraic Number Theory
Exercise Sheet 3

Prof. Dr. Nikita Geldhauser
PD Dr. Maksim Zhykhovich

Winter Semester 2024-25
04.11.2024

Exercise 1. Let A be an integral domain and K the field of fractions of A. Let L and L' be two finite field extension of K. Let B (respectively, B') be the integral closure of A in L (respectively, in L'). Assume that there is a K-morphism of fields $\sigma : L \to L'$. Show that $\sigma(B) \subset B'$.
Deduce that, if L/K is Galois, then B is stable under the action of the Galois group of L/K.

Exercise 2. Let A be an integral domain and $S \subset A$ a multiplicative set. Let K denote the field of fractions of A and let L be a finite field extension of K. Let B denote the integral closure of A in L. Show that the integral closure of $A[S^{-1}]$ in L is $B[S^{-1}]$.

Exercise 3. (1) Let K be a field and let $L = K(\alpha)$ be a separable extension of finite degree n of K, and let $F(X)$ be the minimal polynomial of α over K. Prove the following formula

$$D_K^L(1, \alpha, ..., \alpha^{n-1}) = (-1)^{\frac{1}{2}n(n-1)} N_K^L(F'(\alpha)) \,.$$

(2) Let p be an odd prime number, let $\zeta \in \mathbb{C}$ be a primitive p-th root of unity and let $L = \mathbb{Q}(\zeta)$ be the corresponding cyclotomic field. Compute the dicriminant d_L.
Hint: Use the formula from (1).

Exercise 4. Let $\alpha \in \mathbb{C}$ be a root of $X^3 + 2X + 1$ and let $L = \mathbb{Q}(\alpha)$.
Find a $\mathbb{Z}$-basis of $\mathcal{O}_L$.
Hint: Compute $D_{\mathbb{Q}}^L(1, \alpha, \alpha^2)$.